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General Theory for Cross-Ply Laminated Beams

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We present a general formulation of the elasticity theory of the cross-ply composite laminated beam subjected to various loadings such as axial load, bending moment, shear/bending, and torsion. The formulation is based on the integral equation theory, and a direct approach is employed to obtain the boundary integral equations for the analysis of the laminated beam. The integral equations governing the elasticity problem are directly deduced from the reciprocity theorem, by using the singular solutions of the orthotropic elasticity explicitly derived. The numerical solution is achieved by the boundary element method, which gives, once the traction free boundary conditions and the interfacial continuity conditions are enforced, a linear system of algebraic equations. The theoretical approach does not require any a priori assumption, and it is absolutely general with regard to the section properties. The method of analysis provides efficient computation and accurate solutions for many applications. Four-ply, symmetric cross-ply laminates under uniform axial strain and shear/bending loading are examined in detail. Numerical results are presented and compared with existing data.

Nomenclature

$\boldsymbol{D}$, $\boldsymbol{S}$	= strain operators
$\boldsymbol{D}_n$, $\boldsymbol{S}_n$	= boundary traction operators
$\boldsymbol{E}$, $\boldsymbol{Q}$, $\boldsymbol{G}$	= elasticity matrices
E_{ij}, G_{ij}	= elasticity stiffness coefficients
$\boldsymbol{e}$	= vector of load parameters
$\boldsymbol{e}_n$	= vector of axial and pure bending load parameters
$\boldsymbol{e}_t$	= vector of shear/bending load parameters
$\boldsymbol{f}_j$, $\boldsymbol{b}_j$, f_{3j}	= fundamental solution body forces
l	= laminate length
$\boldsymbol{s}_j$, w_j	= fundamental solution displacements
s_1, s_2, s_3	= displacements in the x_1, x_2, x_3 directions
$\boldsymbol{t}$, t_3	= boundary tractions
$\boldsymbol{t}_j$, t_{3j}	= fundamental solution boundary tractions
$\boldsymbol{t}_n$	= boundary tractions due to axial and pure bending loads
$\boldsymbol{t}_t$	= boundary tractions due to shear/bending loads
x_1, x_2, x_3	= coordinate system for the laminate, with x_3 equal to z
α_1, α_2	= boundary normal direction cosines
Γ_e	= ply section boundary
δ_{ij}	= Kronecker δ
ε, ε_{33}, γ	= strain field
ε_{ij}, γ_{ij}	= strain components
ε_j, ε_{33j}, γ_j	= fundamental solution strains
ε_n	= strain vector due to axial and pure bending loads
ε_t	= strain vector due to shear/bending loads
ν_{ij}	= Poisson's coefficients
σ, σ_{33}, τ	= stress field
σ_{ij}, τ_{ij}	= stress components
σ_j, σ_{33j}, τ_j	= fundamental solution stresses
Ω	= laminate cross section
Ω_e	= ply cross section

Introduction

THE interlaminar stresses can lead to delamination and eventual failure of a multilayered, fiber reinforced composite laminate subjected to static loads. The mismatch in elastic properties between plies may cause complex three-dimensional stress states, showing high gradients in the free edge interlaminar regions, which are not present in homogeneous beams where the de Saint Venant theory can be conveniently applied. Several analysts have computed stress distributions in composite laminates subjected to uniaxial loading or pure bending. In particular, a great variety of approaches have been used to attempt to calculate the interlaminar stresses due to the free edge effect. The first approach where a complete three-dimensional analysis is performed was presented by Pipes and Pagano,[1] who employed the finite difference technique to obtain the solution of the governing elasticity equations. Many solutions obtained by using the finite element method are available.[2–9] These differ from each other in the formulation, the kind of employed elements, and the discretization schemes.

In the literature, analytical solutions of approximate theories are also present. The techniques employed to achieve these solutions include the perturbation method,[10] series solution,[11,12] Lekhnitskii's complex stress potentials coupled with an eigenfunction expansion[13,14] or a polynomial expansion,[15,16] the extension of Reissner's variational principle,[17,18] and the force balance method or the use of equilibrated stress representations coupled with the minimum complementary energy principle.[19–21] The stress distributions obtained by different authors show good agreement between them for sites away from the free edge. However, considerable disagreement exists for points near the free edge location among the various analytical and numerical solutions proposed. Inasmuch as high stress gradients occur near the free edge, the approximate and numerical solutions are not capable of predicting the singularity in the stress field. This is to be expected as a result of a priori assumptions or because the traction boundary conditions of the continuum problem have been transformed into the generalized tractions through equivalent nodal forces. Thus, to obtain accurate stress distributions, it is necessary to use progressive mesh refinement.

In the present paper the stress and strain fields in multilayered cross-ply composite laminates subjected to uniaxial loading, bending moment, torsion, and shear/bending loading are analyzed. The laminated beam consists of prismatic elements having different elastic properties. The generic element, of constant section, is presumed to be homogeneous and orthotropic. The classical de Saint Venant approach is not yet valid because the anisotropy and inhomogeneity may cause complex three-dimensional stress states. Furthermore, the de Saint Venant postulate makes it possible to assume the solution at a certain distance from the extremity zones to be only dependent on the resultant actions on the beam end sections. As we know, these central solutions are linear and homogeneous functions of the force and the moment resultants. Thus, it is important to recover a de Saint Venant solution for anisotropic and inhomogeneous beams considering the basic three dimensionality of the problem. In the present approach stress and strain fields in cross-ply laminated beams are analyzed by extending a method previously presented.[22–25] This method is based on the integral equation theory[26,27] and a direct approach is used to obtain an exact boundary integral equation formulation. By using suitable singular

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particular solutions of the orthotropic elasticity problem, explicitly derived,[22,24,25,28] the integral equations governing the problem are directly obtained by applying the reciprocity theorem. The proposed formulation gives a convenient basis for a numerical solution by the boundary element method (BEM). This approach makes it possible, in the context of the hypotheses presented, to analyze the beam with the widest generalities as regards the shape and the composition of the section and with appreciable computational advantages with respect to other methods that have been proposed. The approach proposed can be applied to laminated beams subjected to arbitrarily distributed loads acting along the beam, whose amplitude is so small that the local effect on the stress state is negligible,[7] such as postulated in the de Saint Venant approach. Some applications to symmetric cross-ply laminates are presented to check the accuracy and the efficiency of the present method.

Definitions

Consider a beam type composite laminate, having a cross section Ω and length l, subjected to axial loading, bending moment, torsion and shear/bending loading. The loads are applied on the beam's terminal sections only. Let the laminate be composed by generally stacked prismatic plies, perfectly bonded at the interface, and let it be referred to a coordinate system x_i, $i = 1, 2, 3$ with the third axis $x_3 = z$ parallel to the beam generatrices (as shown in Fig. 1). The generic ply, having length l and section Ω_e with boundary Γ_e, is assumed homogeneous and orthotropic with respect to the axis system x_i, $i = 1, 2, 3$. Under the preceding assumptions, following Lekhnitskii,[28] the laminate displacement field can be expressed as

$$s_1 = u_1 - \frac{e_1 z^2}{2} + v_1 z - \frac{e_3 z^3}{6} \tag{1a}$$

$$s_2 = u_2 - \frac{e_2 z^2}{2} + v_2 z - \frac{e_4 z^3}{6} \tag{1b}$$

$$w = \varphi + (e_0 + e_1 x_1 + e_2 x_2) z + \frac{(e_3 x_1 + e_4 x_2) z^2}{2} \tag{1c}$$

where e_0, e_1, e_2, e_3, and e_4 are constant all over the laminate section, whereas the functions u_1, u_2, v_1, v_2, and φ depend on x_1 and x_2 only. The constants e_0, e_1, and e_2 are associated to axial loading and pure bending, whereas the constants e_3 and e_4 are connected to shear/bending. The torsion is represented by considering the rigid rotation of the cross section in the functions v_1 and v_2. This displacement field is able to represent the general elasticity solution of a beam type, cross-ply composite laminate under various types of loading characterized by the resultant actions of the stresses that act on the section. The strain field associated with the displacements (1) is given by

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \boldsymbol{Ds} = \boldsymbol{D}(\boldsymbol{u} + z\boldsymbol{v}) = \boldsymbol{\varepsilon}_n + z\boldsymbol{\varepsilon}_t \tag{2a}$$

$$\varepsilon_{33} = e_0 + e_1 x_1 + e_2 x_2 + (e_3 x_1 + e_4 x_2) z = \boldsymbol{X e}_n + z\boldsymbol{X' e}_t \tag{2b}$$

$$\boldsymbol{\gamma} = \begin{bmatrix} \gamma_{31} \\ \gamma_{32} \end{bmatrix} = \boldsymbol{v} + \boldsymbol{S}\varphi \tag{2c}$$

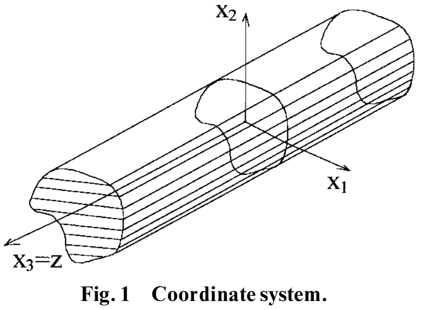

Fig. 1 Coordinate system.

where

$$\boldsymbol{s}^T = [s_1 \quad s_2] \tag{3a}$$

$$\boldsymbol{u}^T = [u_1 \quad u_2] \tag{3b}$$

$$\boldsymbol{v}^T = [v_1 \quad v_2] \tag{3c}$$

$$\boldsymbol{X} = [1 \quad x_1 \quad x_2] \tag{3d}$$

$$\boldsymbol{X'} = [x_1 \quad x_2] \tag{3e}$$

$$e_n^T = [e_0 \quad e_1 \quad e_2] \tag{3f}$$

$$e_t^T = [e_3 \quad e_4] \tag{3g}$$

$$\boldsymbol{D} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix} \tag{3h}$$

$$\boldsymbol{S} = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{bmatrix} \tag{3i}$$

The stresses are

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{12} & E_{22} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \boldsymbol{\varepsilon} + \begin{bmatrix} E_{13} \\ E_{23} \\ 0 \end{bmatrix} \varepsilon_{33}$$

$$= \boldsymbol{E\varepsilon} + \boldsymbol{Q}\varepsilon_{33} = \boldsymbol{E}(\boldsymbol{\varepsilon}_n + z\boldsymbol{\varepsilon}_t) + \boldsymbol{Q}(\boldsymbol{X e}_n + z\boldsymbol{X' e}_t) \tag{4a}$$

$$\sigma_{33} = \boldsymbol{Q}^T\boldsymbol{\varepsilon} + E_{33}\varepsilon_{33} = \boldsymbol{Q}^T(\boldsymbol{\varepsilon}_n + z\boldsymbol{\varepsilon}_t) + E_{33}(\boldsymbol{X e}_n + z\boldsymbol{X' e}_t) \tag{4b}$$

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{31} \\ \tau_{32} \end{bmatrix} = \begin{bmatrix} G_{31} & 0 \\ 0 & G_{32} \end{bmatrix} \begin{bmatrix} \gamma_{31} \\ \gamma_{32} \end{bmatrix} = \boldsymbol{G\gamma} \tag{4c}$$

The equilibrium equations, governing the behavior of each ply, can be expressed as

$$\boldsymbol{D}^T\boldsymbol{\sigma} = \boldsymbol{0} \tag{5a}$$

$$\boldsymbol{S}^T\boldsymbol{\tau} + \frac{\partial \sigma_{33}}{\partial z} = 0 \qquad \text{in} \qquad \Omega_e \tag{5b}$$

$$\boldsymbol{D}_n\boldsymbol{\sigma} = \boldsymbol{t} \tag{6a}$$

$$\boldsymbol{S}_n\boldsymbol{\tau} = t_3 \qquad \text{on} \qquad \Gamma_e \tag{6b}$$

where one has to set

$$\boldsymbol{D}_n = \begin{bmatrix} \alpha_1 & 0 & \alpha_2 \\ 0 & \alpha_2 & \alpha_1 \end{bmatrix} \tag{7a}$$

$$\boldsymbol{S}_n = [\alpha_1 \quad \alpha_2] \tag{7b}$$

$$\boldsymbol{t}^T = [t_1 \quad t_2] \tag{7c}$$

In the preceding relations, α_1 and α_2 are the direction cosines of the outwardly directed normal to the ply section boundary Γ_e whereas t_i indicates the surface forces acting on Γ_e.

Integral Equation Formulation

The generic ply, with section Ω_e having boundary Γ_e, is loaded by the traction system $\boldsymbol{t}$ and t_3 on its lateral surface and by the normal stress σ_{33} and shear stresses $\boldsymbol{\tau}$ on the terminal sections. The traction t_3 and stresses $\boldsymbol{\tau}$ are constant along the z axis, whereas the stress σ_{33} and tractions $\boldsymbol{t}$ are linearly variable along the longitudinal axis z. Let the elementary prismatic solid be subjected to a fictitious system of body forces $\boldsymbol{f}_j$ and let s_{1j}, s_{2j}, and w_j be the components of a particular system of displacements related to $\boldsymbol{f}_j$. Let also $\boldsymbol{\varepsilon}_j$, ε_{33j}, and γ_j be the strains, σ_j, σ_{33j}, and τ_j be the stresses, and $\boldsymbol{t}_j$ and t_{3j} be the boundary tractions, respectively. The equilibrium equations are

$$\boldsymbol{D}^T\sigma_j = -\frac{\partial\tau_j}{\partial z} - \boldsymbol{b}_j \tag{8a}$$

$$\boldsymbol{S}^T\tau_j = -\frac{\partial\sigma_{33j}}{\partial z} - f_{3j} \tag{8b}$$

where

$$\boldsymbol{b}_j^T = [f_{1j} \quad f_{2j}] \tag{9}$$

By applying the reciprocity theorem to the actual ply response and to the particular solution described, considering the first two equilibrium equations and taking into account Eqs. (4), one obtains

$$\int_{\Gamma_e}(\boldsymbol{t}_j^T\boldsymbol{s} - \boldsymbol{s}_j^T\boldsymbol{t})\,\mathrm{d}\Gamma_e + \int_{\Omega_e}\left(\boldsymbol{b}_j + \frac{\partial\tau_j}{\partial z}\right)^T\boldsymbol{s}\,\mathrm{d}\Omega_e$$
$$= \int_{\Omega_e}(\sigma_j^T\varepsilon - \varepsilon_j^T\sigma)\,\mathrm{d}\Omega_e = \int_{\Omega_e}(\varepsilon_{33j}\boldsymbol{Q}^T\varepsilon - \varepsilon_j^T\boldsymbol{Q}\varepsilon_{33})\,\mathrm{d}\Omega_e \tag{10}$$

Substituting the expressions of the displacements $\boldsymbol{s}$ of Eqs. (1) and considering that

$$\int_{\Gamma_e}\boldsymbol{t}_j\,\mathrm{d}\Gamma_e = -\int_{\Omega_e}\left(\boldsymbol{b}_j + \frac{\partial\tau_j}{\partial z}\right)\mathrm{d}\Omega_e \tag{11}$$

Eq. (10) becomes

$$\int_{\Gamma_e}\left[\boldsymbol{t}_j^T(\boldsymbol{u} + z\boldsymbol{v}) - \boldsymbol{s}_j^T\boldsymbol{t}\right]\mathrm{d}\Gamma_e + \int_{\Omega_e}\left(\boldsymbol{b}_j + \frac{\partial\tau_j}{\partial z}\right)^T(\boldsymbol{u} + z\boldsymbol{v})\,\mathrm{d}\Omega_e$$
$$= \int_{\Omega_e}(\varepsilon_{33j}\boldsymbol{Q}^T\varepsilon - \varepsilon_j^T\boldsymbol{Q}\varepsilon_{33})\,\mathrm{d}\Omega_e \tag{12}$$

The tractions $\boldsymbol{t}$ applied along the ply lateral surfaces are the sum of two parts, namely, the tractions due to the axial or pure bending loading, which are constant along the longitudinal axis z, and the tractions concerning shear/bending loading that present a linear variation with respect to z:

$$\boldsymbol{t} = \boldsymbol{t}_n + z\boldsymbol{t}_t \tag{13}$$

Finally, taking into account Eqs. (2), from Eq. (12) one obtains

$$\int_{\Gamma_e}\left[\boldsymbol{t}_j^T(\boldsymbol{u} + z\boldsymbol{v}) - \boldsymbol{s}_j^T(\boldsymbol{t}_n + z\boldsymbol{t}_t)\right]\mathrm{d}\Gamma_e$$
$$+ \int_{\Omega_e}\left(\boldsymbol{b}_j + \frac{\partial\tau_j}{\partial z}\right)^T(\boldsymbol{u} + z\boldsymbol{v})\,\mathrm{d}\Omega_e$$
$$= \int_{\Omega_e}\left[\varepsilon_{33j}\boldsymbol{Q}^T(\varepsilon_n + z\varepsilon_t) - \varepsilon_j^T\boldsymbol{Q}(\boldsymbol{X}\boldsymbol{e}_n + z\boldsymbol{X}'\boldsymbol{e}_t)\right]\mathrm{d}\Omega_e \tag{14}$$

It is evident that Eq. (14) is verified to make the following expressions simultaneously:

$$\int_{\Gamma_e}(\boldsymbol{t}_j^T\boldsymbol{u} - \boldsymbol{s}_j^T\boldsymbol{t}_n)\,\mathrm{d}\Gamma_e + \int_{\Omega_e}\left(\boldsymbol{b}_j + \frac{\partial\tau_j}{\partial z}\right)^T\boldsymbol{u}\,\mathrm{d}\Omega_e$$
$$= \int_{\Omega_e}(\varepsilon_{33j}\boldsymbol{Q}^T\varepsilon_n - \varepsilon_j^T\boldsymbol{Q}\boldsymbol{X}\boldsymbol{e}_n)\,\mathrm{d}\Omega_e \tag{15}$$

$$\int_{\Gamma_e}z(\boldsymbol{t}_j^T\boldsymbol{v} - \boldsymbol{s}_j^T\boldsymbol{t}_t)\,\mathrm{d}\Gamma_e + \int_{\Omega_e}z\left(\boldsymbol{b}_j + \frac{\partial\tau_j}{\partial z}\right)^T\boldsymbol{v}\,\mathrm{d}\Omega_e$$
$$= \int_{\Omega_e}z(\varepsilon_{33j}\boldsymbol{Q}^T\varepsilon_t - \varepsilon_j^T\boldsymbol{Q}\boldsymbol{X}'\boldsymbol{e}_t)\,\mathrm{d}\Omega_e \tag{16}$$

Hence, under the hypotheses that $\boldsymbol{s}_j$ and w_j are constant and linear along z, respectively, deriving Eq. (16) with respect to the third coordinate z, one has

$$\int_{\Gamma_e}(\boldsymbol{t}_j^T\boldsymbol{v} - \boldsymbol{s}_j^T\boldsymbol{t}_t)\,\mathrm{d}\Gamma_e + \int_{\Omega_e}\left(\boldsymbol{b}_j + \frac{\partial\tau_j}{\partial z}\right)^T\boldsymbol{v}\,\mathrm{d}\Omega_e$$
$$= \int_{\Omega_e}(\varepsilon_{33j}\boldsymbol{Q}^T\varepsilon_t - \varepsilon_j^T\boldsymbol{Q}\boldsymbol{X}'\boldsymbol{e}_t)\,\mathrm{d}\Omega_e \tag{17}$$

Equations (15) and (17) in more compact form can be written as

$$\int_{\Gamma_e}(\boldsymbol{t}_j^T[\boldsymbol{u} \quad \boldsymbol{v}] - \boldsymbol{s}_j^T[\boldsymbol{t}_n \quad \boldsymbol{t}_t])\,\mathrm{d}\Gamma_e + \int_{\Omega_e}\left(\boldsymbol{b}_j + \frac{\partial\tau_j}{\partial z}\right)^T[\boldsymbol{u} \quad \boldsymbol{v}]\,\mathrm{d}\Omega_e$$
$$= \int_{\Omega_e}(\varepsilon_{33j}\boldsymbol{Q}^T[\varepsilon_n \quad \varepsilon_t] - \varepsilon_j^T\boldsymbol{Q}[\boldsymbol{X}\boldsymbol{e}_n \quad \boldsymbol{X}'\boldsymbol{e}_t])\,\mathrm{d}\Omega_e \tag{18}$$

From Eq. (18) the first two integral equations can be deduced. Indeed, setting $w_j = 0$ and assuming that the $\boldsymbol{b}_j$, $j = 1, 2$, are the components of a point load applied at the point P_0 along the j direction, one has

$$\boldsymbol{c}_j^T[\boldsymbol{u}(P_0) \quad \boldsymbol{v}(P_0)] + \int_{\Gamma_e}(\boldsymbol{t}_j^T[\boldsymbol{u} \quad \boldsymbol{v}] - \boldsymbol{s}_j^T[\boldsymbol{t}_n \quad \boldsymbol{t}_t])\,\mathrm{d}\Gamma_e$$
$$= -\int_{\Omega_e}\varepsilon_j^T\boldsymbol{Q}[\boldsymbol{X}\boldsymbol{e}_n \quad \boldsymbol{X}'\boldsymbol{e}_t]\,\mathrm{d}\Omega_e = -\int_{\Omega_e}\sigma_{33j}[\boldsymbol{X}\boldsymbol{e}_n \quad \boldsymbol{X}'\boldsymbol{e}_t]\,\mathrm{d}\Omega_e \tag{19}$$

where the $\boldsymbol{c}_j$ vector is given by

$$\boldsymbol{c}_j = \int_{\Omega_e}\boldsymbol{b}_j\,\mathrm{d}\Omega_e = -\int_{\Gamma_e}\boldsymbol{t}_j\,\mathrm{d}\Gamma_e \tag{20}$$

From the third equilibrium equation, taking into account Eqs. (2) and recalling that the $\boldsymbol{s}_j$ displacements are constant along the z direction, one obtains

$$\int_{\Gamma_e}t_{3j}\varphi\,\mathrm{d}\Gamma_e + \int_{\Omega_e}(f_{3j}\varphi + \tau_j^T\boldsymbol{v})\,\mathrm{d}\Omega_e$$
$$= \int_{\Gamma_e}w_j t_3\,\mathrm{d}\Gamma_e + \int_{\Omega_e}w_j\frac{\partial\sigma_{33}}{\partial z}\,\mathrm{d}\Omega_e \tag{21}$$

Taking the derivative of Eq. (21) with respect to z, subtracting it from Eq. (17), and then taking into account the expression of σ_{33} given by Eq. (4b), one obtains

$$\int_{\Gamma_e}\left(\boldsymbol{t}_j^T\boldsymbol{v} - \varphi\frac{\partial t_{3j}}{\partial z} - \boldsymbol{s}_j^T\boldsymbol{t}_t + t_3\frac{\partial w_j}{\partial z}\right)\mathrm{d}\Gamma_e$$
$$+ \int_{\Omega_e}\left(\boldsymbol{b}_j^T\boldsymbol{v} - \varphi\frac{\partial f_{3j}}{\partial z}\right)\mathrm{d}\Omega_e = -\int_{\Omega_e}\sigma_{33j}\frac{\partial\varepsilon_{33}}{\partial z}\,\mathrm{d}\Omega_e \tag{22}$$

Under the hypothesis that $\boldsymbol{b}_j = \boldsymbol{0}$ and that f_{3j} is a concentrated force applied at P_0, Eq. (22) gives the third integral equation

$$c_{33}\varphi(P_0) + \int_{\Gamma_e}\left(\boldsymbol{t}_j^T\boldsymbol{v} - \varphi\frac{\partial t_{3j}}{\partial z} - \boldsymbol{s}_j^T\boldsymbol{t}_t + t_3\frac{\partial w_j}{\partial z}\right)\mathrm{d}\Gamma_e$$
$$= -\int_{\Omega_e}\sigma_{33j}\boldsymbol{X}'\boldsymbol{e}_t\,\mathrm{d}\Omega_e \tag{23}$$

where

$$c_{33} = -\int_{\Omega_e}\frac{\partial f_{3j}}{\partial z}\,\mathrm{d}\Omega_e = \int_{\Gamma_e}\frac{\partial t_{3j}}{\partial z}\,\mathrm{d}\Gamma_e \tag{24}$$

Equations (19) and (23) provide the relations that allow one to obtain, for P_0 belonging to the boundary Γ_e and for $j = 1, 2, 3$, the integral equations linking the displacements and tractions on the ply section boundary. The domain integrals on the right-hand side of the integral equations (19) and (23) can be converted into boundary integrals. Let us consider a particular system of displacements $\bar{\boldsymbol{s}}$ associated with the linear strain ε_{33}. According to Lekhnitskii,[28] the particular solution was obtained by integrating the elastic equilibrium equations of the beam. As a result, indicating with e_5 the unit torsion angle, one has the following displacement field:

$$\bar{s}_1 = -\nu_{31}\left(e_0 + e_1\frac{x_1}{2} + e_2 x_2\right)x_1 + \nu_{32}e_1\frac{x_2^2}{2} - e_1\frac{z^2}{2}$$

$$- \left[e_3\left(\frac{\nu_{31}x_1^2 - \nu_{32}x_2^2}{2}\right) + e_4\nu_{31}x_1x_2 + e_5x_2\right]z - e_3\frac{z^3}{6}$$

$$= \bar{u}_1 - e_1\frac{z^2}{2} + \bar{v}_1 z - e_3\frac{z^3}{6} \tag{25a}$$

$$\bar{s}_2 = -\nu_{32}\left(e_0 + e_1x_1 + e_2\frac{x_2}{2}\right)x_2 + \nu_{31}e_2\frac{x_1^2}{2} - e_2\frac{z^2}{2}$$

$$- \left[e_3\nu_{32}x_1x_2 - e_4\left(\frac{\nu_{31}x_1^2 - \nu_{32}x_2^2}{2}\right) - e_5x_1\right]z - e_4\frac{z^3}{6}$$

$$= \bar{u}_2 - e_2\frac{z^2}{2} + \bar{v}_2 z - e_4\frac{z^3}{6} \tag{25b}$$

$$\bar{w} = (e_0 + e_1x_1 + e_2x_2)z + (e_3x_1 + e_4x_2)\frac{z^2}{2}$$

$$+ e_3\mu_{31}\frac{x_1^3}{6} + e_4\mu_{32}\frac{x_2^3}{6}$$

$$= (e_0 + e_1x_1 + e_2x_2)z + (e_3x_1 + e_4x_2)\frac{z^2}{2} + \bar{\varphi} \tag{25c}$$

where ν_{31} and ν_{32} are Poisson's coefficients and μ_{31} and μ_{32} are given by

$$\mu_{3i} = \frac{(E_{13} + G_{31})\nu_{31} + (E_{23} + G_{32})\nu_{32} - E_{33}}{G_{3i}} \tag{26}$$

From Eqs. (19) and (23), because of $\bar{\sigma}$ is identically null, one has

$$\boldsymbol{c}_j^T[\bar{\boldsymbol{u}}(P_0) \quad \bar{\boldsymbol{v}}(P_0)] + \int_{\Gamma_e} \boldsymbol{t}_j^T[\bar{\boldsymbol{u}} \quad \bar{\boldsymbol{v}}]\,\mathrm{d}\Gamma_e$$

$$= -\int_{\Omega_e} \sigma_{33j}[\boldsymbol{X}\boldsymbol{e}_n \quad \boldsymbol{X}'\boldsymbol{e}_t]\,\mathrm{d}\Omega_e \tag{27}$$

$$c_{33}\bar{\varphi}(P_0) + \int_{\Gamma_e}\left[\boldsymbol{t}_j^T\bar{\boldsymbol{v}} - \bar{\varphi}\frac{\partial t_{3j}}{\partial z} + \bar{t}_3\frac{\partial w_j}{\partial z}\right]\mathrm{d}\Gamma_e$$

$$= -\int_{\Omega_e} \sigma_{33j}\boldsymbol{X}'\boldsymbol{e}_t\,\mathrm{d}\Omega_e \tag{28}$$

Finally, subtracting Eq. (27) from Eq. (19) and Eq. (28) from Eq. (23), one obtains

$$\boldsymbol{c}_j^T[\boldsymbol{u}(P_0) \quad \boldsymbol{v}(P_0)] + \int_{\Gamma_e}\left(\boldsymbol{t}_j^T[\boldsymbol{u} \quad \boldsymbol{v}] - \boldsymbol{s}_j^T[\boldsymbol{t}_n \quad \boldsymbol{t}_t]\right)\mathrm{d}\Gamma_e$$

$$= \boldsymbol{c}_j^T[\bar{\boldsymbol{u}}(P_0) \quad \bar{\boldsymbol{v}}(P_0)] + \int_{\Gamma_e} \boldsymbol{t}_j^T[\bar{\boldsymbol{u}} \quad \bar{\boldsymbol{v}}]\,\mathrm{d}\Gamma_e \tag{29}$$

$$c_{33}\varphi(P_0) + \int_{\Gamma_e}\left(\boldsymbol{t}_j^T\boldsymbol{v} - \varphi\frac{\partial t_{3j}}{\partial z} - \boldsymbol{s}_j^T\boldsymbol{t}_t + t_3\frac{\partial w_j}{\partial z}\right)\mathrm{d}\Gamma_e$$

$$= c_{33}\bar{\varphi}(P_0) + \int_{\Gamma_e}\left(\boldsymbol{t}_j^T\bar{\boldsymbol{v}} - \bar{\varphi}\frac{\partial t_{3j}}{\partial z} + \bar{t}_3\frac{\partial w_j}{\partial z}\right)\mathrm{d}\Gamma_e \tag{30}$$

which constitute the boundary integral representation of the problem dealt with. Equations (29) and (30) give the integral equations coupling the tractions and the displacements on the boundary of the generic ply. These equations, together with the prescribed boundary data and the interfacial conditions, are the fundamental relations for the solution of the problem by the BEM. When the ply section boundary is discretized by boundary elements, the unknowns involved in Eq. (29) are, in general, four for each nodal point, namely, $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{t}_n$, and $\boldsymbol{t}_t$. On the other hand, for each node, that is for P_0 coincident with the nodal point, one can write term to term four equations, P_0 being common to the boundaries of the contiguous plies. Once the fundamental solutions are known, taking into account the prescribed boundary data and enforcing the interface continuity conditions, a set of linear algebraic equations is obtained, the solution of which provides the displacements and the tractions along the x_1 and x_2 directions on the boundary of each ply. Then from Eq. (30) one can determine the other ply section boundary unknowns, i.e., the function φ and the tractions t_3 directed along the z axis.

Fundamental Solutions

The fundamental solutions, i.e., the orthotropic elastic responses in infinite domain due to point loads acting at P_0, depend on the roots of the following characteristic equation[22,28,29]:

$$C_{11}\lambda^2 - 2(C_{12} + C_{33}/2)\lambda + C_{22} = 0 \tag{31}$$

where the coefficients C_{rs} are the elastic coefficients of the ply

$$[C_{rs}] = \boldsymbol{E}^{-1} \tag{32}$$

Assuming that the roots λ_i of Eq. (31) are distinct and positive in sign, as generally happens, the first two fundamental solutions $\boldsymbol{s}_j$ are

$$\begin{bmatrix} s_{1j} \\ s_{2j} \end{bmatrix} = \begin{bmatrix} \varphi_{1j}B_{11} & \varphi_{2j}B_{12} \\ \psi_{1j}B_{21} & \psi_{2j}B_{22} \end{bmatrix}\begin{bmatrix} A_{1j} \\ A_{2j} \end{bmatrix} \tag{33}$$

in which

$$\varphi_{i1}(P, P_0) = -(\ln r_i)/2\pi \tag{34a}$$

$$\psi_{i1}(P, P_0) = -[\tan^{-1}(\sqrt{\lambda_i}y/x)]/2\pi\sqrt{\lambda_i} \tag{34b}$$

$$\varphi_{i2}(P, P_0) = -[\tan^{-1}(\sqrt{\lambda_i}y/x)]\sqrt{\lambda_i}/2\pi \tag{34c}$$

$$\psi_{i2}(P, P_0) = (\ln r_i)/2\pi \tag{34d}$$

where

$$x = x_1(P) - x_1(P_0) \tag{35a}$$

$$y = x_2(P) - x_2(P_0) \tag{35b}$$

$$r_i = \sqrt{x^2 + \lambda_i y^2} \tag{35c}$$

where $x_i(P)$ are the coordinates of the field point P and $x_i(P_0)$ are the location coordinates of the point force. Moreover, one has to set

$$B_{1i} = C_{11} - C_{12}/\lambda_i \tag{36a}$$

$$B_{2i} = C_{12} - C_{22}/\lambda_i \tag{36b}$$

The stresses σ_j, associated to these fundamental solutions, are given by

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}_j = \begin{bmatrix} \dfrac{\partial\varphi_{1j}}{\partial x_1} & \dfrac{\partial\varphi_{2j}}{\partial x_1} \\ -\dfrac{\partial\psi_{1j}}{\lambda_1\partial x_2} & -\dfrac{\partial\psi_{2j}}{\lambda_2\partial x_2} \\ \dfrac{\partial\varphi_{1j}}{\lambda_1\partial x_2} & \dfrac{\partial\varphi_{2j}}{\lambda_2\partial x_2} \end{bmatrix}\begin{bmatrix} A_{1j} \\ A_{2j} \end{bmatrix} \tag{37}$$

The A_{ij} constants are provided by

$$\boldsymbol{A}_1 = \frac{\lambda_1\lambda_2}{C_{22}(\lambda_2 - \lambda_1)}\begin{bmatrix} B_{22}\sqrt{\lambda_1} \\ -B_{21}\sqrt{\lambda_2} \end{bmatrix} \tag{38a}$$

$$\boldsymbol{A}_2 = \frac{\lambda_1\lambda_2}{C_{11}(\lambda_1 - \lambda_2)}\begin{bmatrix} -B_{12}/\sqrt{\lambda_1} \\ B_{11}/\sqrt{\lambda_2} \end{bmatrix} \tag{38b}$$

The third fundamental singular solution of the problem was explicitly derived by using Lekhnitskii's[28] stress functions. This solution is given by the following relationship:

$$\begin{bmatrix} s_{13} \\ s_{23} \\ w_3 \end{bmatrix} = \begin{bmatrix} \varphi_{13} B_{11} & \varphi_{23} B_{12} & 0 \\ \psi_{13} B_{21} & \psi_{23} B_{22} & 0 \\ 0 & 0 & x_3 \varphi_{33} \end{bmatrix} \begin{bmatrix} A_{13} \\ A_{23} \\ A_{33} \end{bmatrix} \tag{39}$$

where

$$\varphi_{i3} = x\left(\sqrt{\lambda_i}[\ln(r_i) - 1] - \lambda_i[\ln(r_3) - 1]/\sqrt{\lambda_3}\right) + \lambda_i y \tan^{-1}\left[(\sqrt{\lambda_3} - \sqrt{\lambda_i})xy\Big/(x^2 + \sqrt{\lambda_i\lambda_3}y^2)\right] \tag{40a}$$

$$\psi_{i3} = y\left(\sqrt{\lambda_i}[\ln(r_i) - 1] - \sqrt{\lambda_3}[\ln(r_3) - 1]\right) - x \tan^{-1}\left[(\sqrt{\lambda_3} - \sqrt{\lambda_i})xy\Big/(x^2 + \sqrt{\lambda_i\lambda_3}y^2)\right] \tag{40b}$$

$$\varphi_{33} = \ln(r_3) \tag{40c}$$

and where

$$\lambda_3 = G_{31}/G_{32} \tag{41a}$$

$$r_3 = \sqrt{x^2 + \lambda_3 y^2} \tag{41b}$$

The stresses are

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}_3 = \begin{bmatrix} \sqrt{\lambda_1}\ln(r_1) - \sqrt{\lambda_3}\ln(r_3) & 0 & 0 \\ \ln(r_3)/\sqrt{\lambda_3} - \ln(r_1)/\sqrt{\lambda_1} & 0 & 0 \\ \tan^{-1}\left[(\sqrt{\lambda_3} - \sqrt{\lambda_1})xy\Big/(x^2 + \sqrt{\lambda_1\lambda_3}y^2)\right] & 0 & 0 \end{bmatrix} \begin{bmatrix} A_{13} \\ A_{23} \\ A_{33} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & \sqrt{\lambda_2}\ln(r_2) - \sqrt{\lambda_3}\ln(r_3) & -G_{31}\ln(r_3) \\ 0 & \ln(r_3)/\sqrt{\lambda_3} - \ln(r_2)/\sqrt{\lambda_2} & -G_{32}\ln(r_3) \\ 0 & \tan^{-1}\left[(\sqrt{\lambda_3} - \sqrt{\lambda_2})xy\Big/(x^2 + \sqrt{\lambda_2\lambda_3}y^2)\right] & 0 \end{bmatrix} \times \begin{bmatrix} A_{13} \\ A_{23} \\ A_{33} \end{bmatrix} \tag{42a}$$

$$\begin{bmatrix} \tau_{31} \\ \tau_{32} \end{bmatrix}_3 = \begin{bmatrix} G_{31} & 0 \\ 0 & G_{32} \end{bmatrix} \begin{bmatrix} x\big/r_3^2 \\ \lambda_3 y\big/r_3^2 \end{bmatrix} x_3 A_{33} \tag{42b}$$

For this fundamental solution, the coefficients A_{i3} need to be calculated from the following relation:

$$\begin{bmatrix} A_{13} \\ A_{23} \\ A_{33} \end{bmatrix} = \frac{\sqrt{\lambda_3/(G_{31}G_{32})}}{2\pi(\lambda_1 - \lambda_2)} \begin{bmatrix} \dfrac{\lambda_1}{(\lambda_3 - \lambda_1)} & \dfrac{-\lambda_1\lambda_2}{(\lambda_3 - \lambda_1)} & 0 \\ \dfrac{-\lambda_2}{(\lambda_3 - \lambda_2)} & \dfrac{\lambda_1\lambda_2}{(\lambda_3 - \lambda_2)} & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \dfrac{C_{11}G_{31} + C_{12}G_{32} + \nu_{13}}{C_{11}} \\ \dfrac{C_{12}G_{31} + C_{22}G_{32} + \nu_{23}}{C_{22}} \\ \dfrac{(\lambda_2 - \lambda_1)}{\sqrt{\lambda_3}} \end{bmatrix} \tag{43}$$

The fundamental solutions in an infinite orthotropic domain are calculated for a unit point force at the location P_0. It can be proved that, whereas the coefficient c_{ij} of the Eq. (20) is equal to δ_{ij} when P_0 is in the interior domain Ω_e, the value of c_{ij} is equal to $\delta_{ij}/2$ when P_0 is on the smooth boundary Γ_e.

Applications

In this section some applications are presented to illustrate the formulation proposed. The examples refer to two cross-ply laminate configurations, namely, $(0/90)_s$ and $(90/0)_s$, that have been widely investigated in literature and for which solutions are available for comparison. The geometrical properties of the investigated laminates are shown in Fig. 2 and the material elastic properties in gigapascal are set as follows:

$$E_{LL} = 137.9 \qquad E_{TT} = E_{SS} = 14.5$$
$$G_{LT} = G_{LS} = G_{TS} = 5.9 \qquad \nu_{LT} = \nu_{LS} = \nu_{TS} = 0.21 \tag{44}$$

where the subscripts L, T, and S refer to along fiber, transverse, and thickness directions, respectively. Owing to the structural symmetry only a quarter of the laminate section has been considered in the analysis, and for each ply the discretization shown in Fig. 3 has been used. The integrals have been calculated by Gauss quadrature with linear interpolation of the unknown boundary data. Suitable numerical techniques have been used to take into account the kernel singularities.[30] At the corner points the coefficients c_{ij} are numerically obtained by calculating the boundary integral of Eq. (20). The assumption of symmetric stress tensor requires $\sigma_{21} = 0$ at the free edge location $x_2 = h$, so as to satisfy the local momentum equilibrium. To satisfy this free edge condition, the two equations written for a point load applied at the free edge interlaminar corner and directed along the x_1 axis are substituted by a linear combination obtained by summing them.

Figures 4 and 5 show the interlaminar stresses σ_{22} and σ_{21}, respectively, at the interface $x_2 = h$, for the two laminates subjected to axial extension. The stress distributions obtained show good agreement with those available in the literature for points away from the free edge. However, considerable differences exist for points near the free edge location. This is to be expected as a result of a priori assumptions for approximate solutions or because the boundary traction conditions of the continuum problem have been transformed into conditions on the generalized data. In the present solution (as shown in Fig. 5), the interlaminar shear stress σ_{21} goes to zero at the free edge as must be in order to agree with the traction-free boundary condition. However, even if the shear stress σ_{21} vanishes at the free edge, the stress distribution near the free edge (Fig. 5) increases rapidly toward large values, which suggests a significant stress concentration.

Figures 6, 7, and 8 show the interlaminar stresses σ_{21}, σ_{22}, and σ_{23}, respectively, at the interface $x_2 = h$, for the two laminates subjected to a shear loading along the x_2 axis. Figures 6 and 7 suggest that the

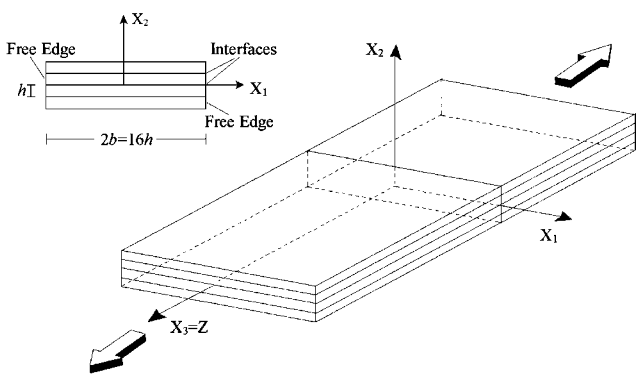

Fig. 2 Laminate configuration.

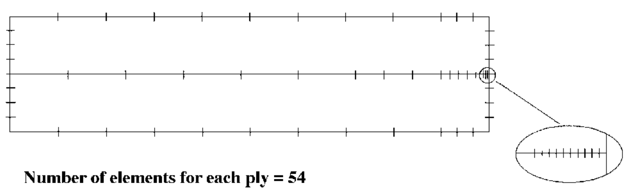

Fig. 3 Boundary element method discretization for a quarter of the beam section.

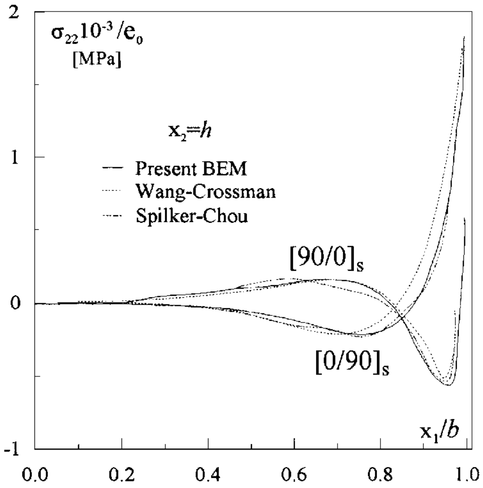

Fig. 4 Stress σ_{22} distribution for [0/90]$_s$ and [90/0]$_s$ laminates under axial extension.

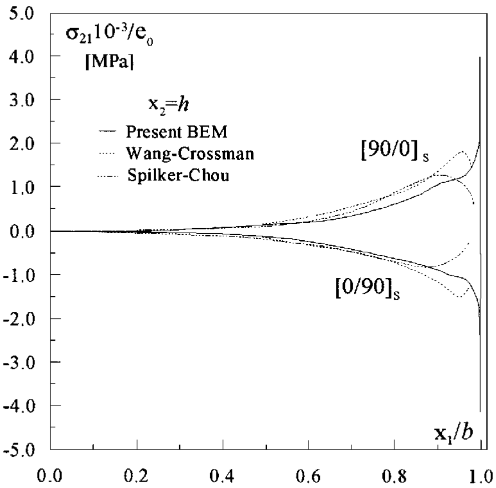

Fig. 5 Stress σ_{21} distribution for [0/90]$_s$ and [90/0]$_s$ laminates under axial extension.

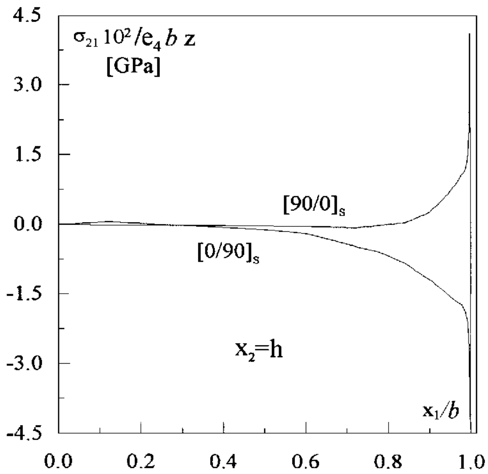

Fig. 6 Stress σ_{21} distribution for [90/0]$_s$ and [0/90]$_s$ laminates at the generic longitudinal coordinate *z* under shear/bending loading.

interlaminar stress σ_{21} and peeling stress σ_{22} distributions along the interface have the same trend as those found for laminates under uniform axial extension for both laminates considered. It is also seen that a rapid change of gradients occurs near the free edge for the σ_{23} interlaminar shear stress in composite laminates under shear/bending. By summarizing the results, the solutions indicate that complex stress states with significant stress concentrations near the free edge are present in cross-ply laminates subjected to various

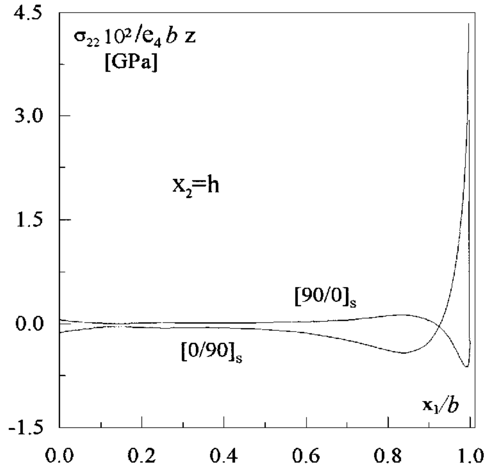

Fig. 7 Stress σ_{22} distribution for [90/0]$_s$ and [0/90]$_s$ laminates at the generic longitudinal coordinate *z* under shear/bending loading.

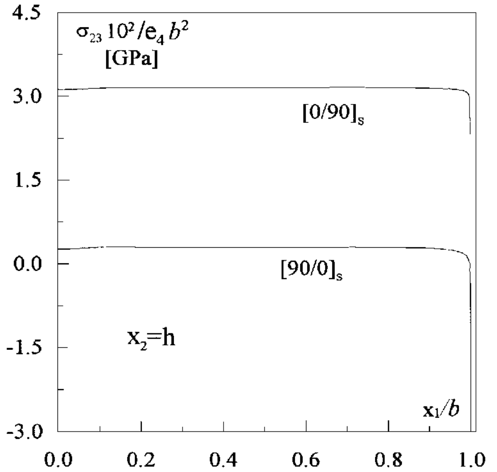

Fig. 8 Stress σ_{23} distribution for [90/0]$_s$ and [0/90]$_s$ laminates under shear/bending loading.

loading. The present analysis appears to confirm the existence of stress singularities in the stress field at the free edge as pointed out by other authors.[5,6,8,13] This problem is the subject of in progress analyses.

Conclusions

The formulation proposed provides the elasticity solution for the multilayered, beam type composite laminate under various loadings applied only on the terminal sections. The problem is exactly formulated in terms of integral equations, and the solution is numerically achieved by using the BEM. All of the elasticity relations, that is, equilibrium equations, strain compatibility, traction boundary conditions, and ply interface continuity conditions, are exactly satisfied in the formulation, which therefore proves consistent. The numerical solution provided by the BEM thus tends to the solution of the continuous model with the mesh refinement. The present formulation is a very interesting approach to the problem both from the theoretical and the computational point of view. It allows the analysis of structural components of composite material working in the context of no a priori assumptions. Four-ply, symmetric cross-ply laminates, subjected to axial and shear/bending loads have been investigated and the results compared, when available, with those obtained using other solution techniques. The interlaminar stress pattern near the free edge, which was obtained without appealing to successive mesh refinements, highlights the efficacy of the method.

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